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GOSSIPS AND TELEPHONES

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The following problem has circulated lately among mathematicians. Other solutions have been given independently by R.T. Bumby and by Hajnal et al. [1].

The problem. There are n ladies, and each of them knows some item of gossip not known to the others. They communicate by telephone, and whenever one lady calls another, they tell each other all that they know at that time. How many calls are required before each gossip knows everything?

Answer. Let $f(n)$ be the minimum number of calls needed for n people. It is easily shown that $f(1) = 0$, $f(2) = 1$, $f(3) = 3$ and $f(4) = 4$. For $n > 4$, $2n-4$ calls are sufficient according to the following procedure: one of four "chief" gossips first calls each of the remaining $n-4$ gossips, then the four learn each other's (and hence everyone's) information in 4 calls (as $f(4) = 4$), and finally one of the four chiefs calls each of the other $n-4$ gossips.

Theorem 1. $f(n) = 2n-4$ *for* $n > 4$.

Proof. Suppose to the contrary that for some $n > 4$, $f(n) \leq 2n-5$. Let m be the least such n and let S be any calling arrangement among m gossips requiring at most $2m-5$ calls. We will obtain a contradiction based upon the following lemma.

Lemma 1. *No gossip may hear her own information from another in S.*

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Proof. If gossip G can hear her own information in S, there is a sequence of calls $(G\text{-}G_1)(G_1\text{-}G_2)$ ... $(G_r\text{-}G)$, listed in temporal order. By omitting gossip G, we obtain from S a new arrangement T of calls among $m-1$ gossips as follows:

Omit from S calls $(G\text{-}G_1)$ and $(G_r\text{-}G)$. In addition, for each gossip P who makes a call $(P\text{-}G)$ (other than $(G\text{-}G_1)$ and $(G_r\text{-}G)$ of the above sequence) in S, let the G_i's transmit P's information in T as follows: Let t be the least i such that $(G_i\text{-}G_{i+1})$ occurs after $(P\text{-}G)$ in S, if such a call exists, and r otherwise. Replace $(P\text{-}G)$ in S by a call $(P\text{-}G_t)$ in T, preserving the temporal ordering of the calls. The arrangement T will contain at most $2(m-1)-5$ calls and it is easily verified that all ladies learn all of the gossip in T if they do so in S. The lemma follows by minimality of m.

Conclusion of the proof of Theorem 1. By Lemma 1, a call is either the final call for both parties to the call in S or it is the final call for neither (for after receiving gossip B's final call, gossip A knows everything and would violate the Lemma by later calling someone else). Also, a call is either the initial call for both parties or for neither (otherwise, if A makes her first call to B after B calls C, then information from C would propagate with A's until it came back to C, contradicting Lemma 1).

Thus initial and final calls account for m calls *. Let the remaining calls be described as intermediate calls and let I be a graph with m nodes representing gossips and edges representing intermediate calls. Since there are at most $m-5$ intermediate calls and since $m-1$ edges are needed for a graph of m nodes to be connected, the graph I must contain at least five disjoint connected components. Information from a given gossip G can propagate into only two components (hers and her initial caller's) before any final calls are made. Similarly, after the initial calls have been made, information may be transmitted to her through the calls of only two components (hers and her final caller's). Thus the calls of at least two components of intermediate calls play no part in propagating her information or informing her. For a gossip G, let $c(G)$ be the number of calls which are not used in transmitting information to her or from her.

At least $n-1$ calls are required to inform a given gossip completely and $n-1$ are required to transmit her information. By Lemma 1 the only

* Clearly a call cannot be both initial and final.

calls which can do both are those that she herself takes part in – otherwise she would hear her information from another. Thus at least $2n-2-v(G)$ calls are required to convey gossip G's information and inform her, where $v(G)$ is the number of calls in which she participates. From $2n-5 \geq 2n-2-v(G)+c(G)$ we conclude that $v(G) \geq 3+c(G) \geq 3$. Since $v(G) \geq 3$ for each gossip G, every connected component of intermediate calls contains an edge – otherwise some gossip would make only an initial and a final call. According to the previous paragraph, the calls of at least two components are not used in transmitting information to or from G. Hence $c(G) \geq 2$ and $v(G) \geq 5$ for all gossips G, resulting in more than $2n$ calls altogether.

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Reference

[1] A. Hajnal, E.C. Milner and E. Szemerédi, A cure for the telephone disease, Res. Paper No. 106, Univ. Calgary, Calgary, Alberta, Canada, January, 1971.